Nonlinear systems and their applications to the regularized Boussinesq system

Mudhir A. Abdul Hussain
Department of Mathematics ,College of Education,
University of Basrah,
e-mail: mud_abd@yahoo.com

ISSN –1817 –2695
((Received 1/11/2006 , Accepted 15/11/2007))

Abstract:

In this paper, bifurcation analysis of the cubic nonlinear system has been studied. It is shown that there is a regular solution in some domains of parameters of a given system, also the Discriminant set of a certain system was found. The existence and stability of periodic solutions of the regularized Boussinesq system were shown.

Key Words: Local bifurcation theory, Local scheme of Lyapunov-Schmidt, Regularized Boussinesq system.

1. Introduction:

It is known that many of the nonlinear problems in mathematics and physics can be written in the form of operator equation:

$$f(x,\lambda)=b, \quad x\in O\subset X, \; b\in Y, \; \lambda\in R^n. \quad \ldots (1.1)$$

in which f is a smooth Fredholm map of index zero, X, Y Banach spaces and O open subset of X. For these problems, we can use method of reduction to finite dimensional equation [1],

$$\Theta(\xi,\lambda)=\beta , \quad \xi\in M, \quad \beta\in N, \quad \ldots(1.2)$$

where M and N are smooth finite dimensional manifolds.Passage from equation (1.1) into equation (1.2) (variant locally scheme of Lyapunov –Schmidt) with the conditions, that equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, *etc*…) according to [2],[6],[7],[8].In the theory of polynomials and holomorphics maps [3] used the term of Discriminant set which is similar to the term used in the modern theory of bifurcation. The study of bifurcation solutions of equation (1.1) is equivalent to the study the bifurcation solutions of equation (1.2). In the method of finite dimensional reduction it is necessary to study the normal forms (Nonlinearities) to find the bifurcation solutions of equation (1.1) which is reduced into equation (1.2) as in this paper.

Definition 1.1 A continuous linear operator $A: E \rightarrow F$, (E, F are Banach Spaces) is called Fredholm iff $\dim(Ker\ A)<\infty$, $\dim(CoKer\ A)<\infty$. The number $\dim(Ker\ A)-\dim(CoKer\ A)$ is called Fredholm index of operator A.

Definition 1.2 The nonlinear map $f: \Omega \rightarrow F$, where Ω is an open subset of E, is called Fredholm, if $\frac{\partial f}{\partial x}(x)$ is a Fredholm operator, $x \in \Omega$. The index of f is the same index of $\frac{\partial f}{\partial x}(x)$.

Definition 1.3 The set of all λ in which equation (1.1) has degenerate solution in O is called the Discriminant set.

2. Cubic Nonlinearities:

In the Dynamical systems and other parts of Mathematics (Control theory) it is useful to study the bifurcation analysis of some nonlinear systems that appear in the Mathematical and Physical problems, these nonlinear systems allow an important connection to be made between the existence of solutions and their stability.The goal is to determine the Discriminant set (bifurcation diagram) of the following system,

$$x^2+2y^2-q_1y+q_2=0,$$
$$y^3+2x^2y+\beta_1x^2+\alpha_2y=0. \qquad (2.1)$$
$$where,\quad x,y>0,\ \ are\ reals\ and\ q_1,q_2,\beta_1,\alpha_2\in R.$$

and then apply the results to the regularized Boussinesq system which model waves in a horizontal water channel in both directions. Changes in the parameters of system (2.1) will lead into changes in the qualitative structure of the solutions of system (2.1). These changes give rise to the bifurcation solutions from the bifurcation point. Thus, when $\beta_1=0$, it is easy to determine the Discriminant set. The useful case is to determine the Discriminant set when $\beta_1\neq 0$ because in the applications the value of β_1 need not to be equal to zero. In both cases the following results has been stated:

Theorem 2.1 If $\beta_1=0$, then Discriminant set of system (2.1) can be described in the parameter equation,

$$q_2=\frac{1}{6}(q_1^2+3\alpha_2).$$

Proof: Since, the solutions of system (2.1) are degenerate on the line $y=\frac{1}{3}q_1$ with $x > 0$, it follows that the Discriminant set of system (2.1) has the following parameterization,

$$q_2=y^2-x^2,$$
$$q_1=3y,$$
$$\alpha_2=-2x^2-y^2.$$

and then from equations above the set of all $\lambda=(q_1,q_2,\alpha_2)$ satisfy the parameter equation

$$q_2=\frac{1}{6}(q_1^2+3\alpha_2). \square$$

Theorem 2.2 If $\beta_1=0$, then for $\alpha_2\geq 0$ there is no regular solution of system (2.1) and for $\alpha_2<0$ there is regular solutions of system (2.1) in some domains of parameters.

Proof: Suppose that S is the set of solutions of system (2.1) and S_1, S_2 be two sets such that,

$$S_1=\{(x,y):\ x^2+2y^2-q_1y+q_2=0\},$$
$$S_2=\{(x,y):\ 2x^2+y^2+\alpha_2=0\}.$$

then $S=S_1\bigcap S_2$. Since $\beta_1=0$, then the second equation of system (2.1) has no solutions for all $\alpha_2\geq 0$ implies that $S_2=\{\varphi\}$ and hence $S=\{\varphi\}$. It is mean that system (2.1) has no solutions for $\alpha_2\geq 0$. When $\alpha_2<0$ the sets S_1 and S_2 are an ellipse. It is known that two ellipse are intersects in 2 , 4 distinct points or not. Since $\alpha_2<0$ implies that $\alpha_2=-m,\ m>0.$ From system (2.1) we have that $y=\frac{1}{3}(q_1\pm\sqrt{q_1^2-6q_2+3\alpha_2})$ and then y have two real values when $q_2<\frac{1}{6}(q_1^2+3\alpha_2)$ and one real value when $q_2=\frac{1}{6}(q_1^2+3\alpha_2)$. From the second equation of system (2.1) we have that the value of x is real only when $0<y<\sqrt{m}$ and hence the sets S_1 and S_2 are intersects in four points when $q_2<\frac{1}{6}(q_1^2+3\alpha_2)$, $0<y<\sqrt{m}$ and in two points when $q_2=\frac{1}{6}(q_1^2+3\alpha_2), 0<y<\sqrt{m}$.

Otherwise, the system has no distinct solutions. Thus, in some domains of parameters system (2.1) has a solution for $\alpha_2 < 0$. □

In many applications, the Discriminant set can be solve by finding a relationship between the parameters and variables given in the problem, but in some problems there is a difficulty for finding this parameterization. The second way for finding the Discriminant set is by finding the parameter equation, that is; equation of the form

$$h(\delta) = 0, \quad \delta = (\lambda_1, \lambda_2, ..., \lambda_n) \in R^n.$$

where $h : R^n \to R$ is a map and $\lambda_1, \lambda_2, ..., \lambda_n$ are parameters. Thus, to determine the Discriminant set of the system (1) when $\beta_1 \neq 0$ it is convenient to find the parameter equation of the form,

$$h_1(q_1, q_2, \beta_1, \alpha_2) = 0, \quad h_1 : R^4 \to R \text{ is a map}$$

such that the set of all $\lambda = (q_1, q_2, \beta_1, \alpha_2)$ in which system (2.1) has degenerate solution will be satisfying the equation $h_1(\lambda) = 0$. Thus, from the above notations the following result was found.

Theorem 2.3 If $\beta_1 \neq 0$, then Discriminant set of system (2.1) can be described in the parameter equation,

$$h_1(\lambda) = 0, \quad \lambda = (q_1, q_2, \beta_1, \alpha_2), \ldots \quad (2.2)$$

$$h_1(\lambda) = a\,G^3 + 2(2b + q_1 a)G^2 H + (2b\beta_1 - 5c)GH^2 - G^2 H^2 + \beta_1(2c - q_2 a)H^3.$$

where,

$$G = 2q_2 a - 9c - \alpha_2 a - q_1 \beta_1 a,$$

$$H = 9b + 4q_1 a - 4\beta_1 a,$$

$$a = 2q_1 + \frac{5}{2}\beta_1, \qquad b = 2(\beta_1^2 - 2q_2 + \alpha_2),$$

$$c = -\beta_1(\frac{1}{2}(\alpha_2 + q_1\beta_1) + 2q_2).$$

The above theorem can be proved by solving the following three equations,

$$x^2 + 2y^2 - q_1 y + q_2 = 0,$$

$$y^3 + 2x^2 y + \beta_1 x^2 + \alpha_2 y = 0.$$

$$2x^2 - 5y^2 + 2(q_1 - 2\beta_1)y + \alpha_2 + q_1\beta_1 = 0.$$

in terms of $q_1, q_2, \beta_1, \alpha_2$. To study the Discriminant set of system (2.1) when $\beta_1 \neq 0$ it is convenient to consider equation (2.2) in the space of parameters. The sections of Discriminant set in some planes was described in the figures (1),(2),(3) and (4). The figures have been drawn by using Maple 9 .

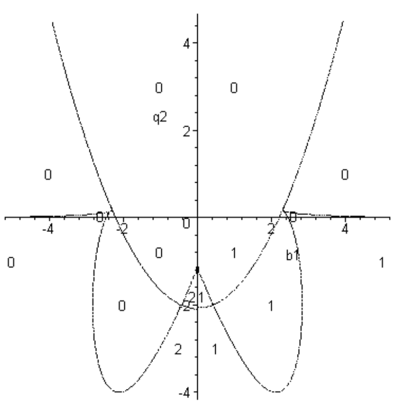

Fig. (1)

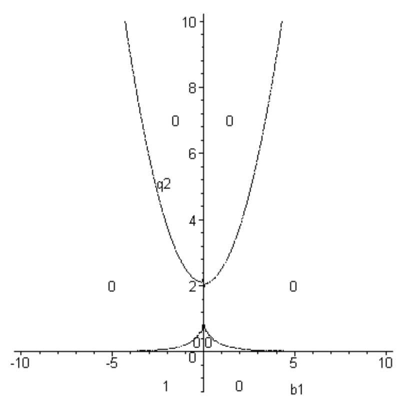

Fig. (2)

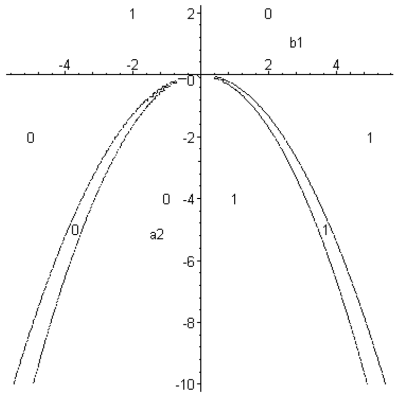

Fig. (3)

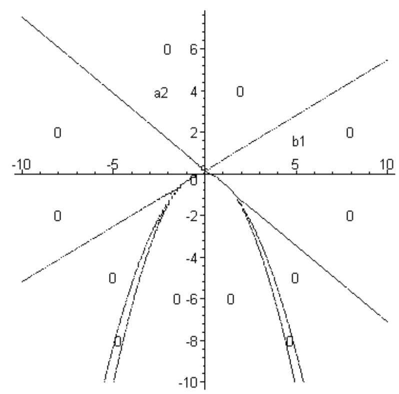

Fig. (4)

Figure (1) describes the Discriminant set in the $\beta_1 q_2$-plane when q_1=0.001 and α_2= -2. The number of positive regular solutions in every region was found. In figure (2) the Discriminant set has been found for q_1=0.01 and α_2= 2 in the $\beta_1 q_2$-plane which is show that there is only one positive regular solution of system (2.1). Figure (3) describes the Discriminant set in the $\beta_1\alpha_2$-plane for q_1=0.1 and q_2= -0.05. The number of positive regular solutions has been found in every domain. Figure (4) describes the Discriminant set in the $\beta_1\alpha_2$-plane when q_1=0.1 and q_2= 0.05 which is show that there is no regular solutions of system (2.1). Note that in figures (1),(2) and (3) there are symmetric domains with different numbers of regular solutions, this is because only positive regular solutions has been found. From above notations the existence and stability solutions of system (2.1) in some domains of parameters has been proved. On the other hand, one can take another sections in the plane of parameters to show the existence of solutions of system (2.1).

3. Application to the regularized Boussinesq system:

Consider the regularized Boussinesq system,

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x - \frac{1}{6}\eta_{xxt} = 0, \\ &u_t + \eta_x + u u_x - \frac{1}{6} u_{xxt} = 0. \end{aligned} \qquad \dots (3.1)$$

which describes approximately two-dimensional propagation of surface waves in a uniform horizontal channel of length L filled with an irrotational, incompressible, invisid fluid which in its undisturbed state has depth h. The non-dimensional variables η (x, t) and u (x, t) represent, respectively, the deviation of the water surface from its undisturbed position and the horizontal velocity at water level $\sqrt{\frac{2}{3}}$ h. The two-modes of bifurcation periodic traveling-wave solution in the form $\xi = r x - k t$, where ($v = k/r$ - is the speed of propagation wave) was studied. In this case system (3.1) has the form,

$$\begin{aligned} &\frac{k r^2}{6}\eta'' + r u - k\eta + r\eta u = c_1, \\ &\frac{k r^2}{6} u'' - k u + r\eta + \frac{r}{2} u^2 = c_2. \end{aligned} \qquad \dots (3.2)$$

where η (x, t) = η (ξ) and u (x, t) = $u(\xi)$. For simply, the values of c_1 and c_2 was chosen to be equal to zero. It is easy to check that system (3.2) can be written as a single equation,

$$u^{////} - \frac{12}{r^2}u^{//} + \frac{12}{kr}uu^{//} + \frac{6}{kr}(u^{/})^2 - \frac{36(r - \frac{k^2}{r})}{k^2 r^3}u - \frac{54}{kr^3}u^2 + \frac{18}{k^2 r^2}u^3 = 0. \quad \dots (3.3)$$

The purpose is to study the bifurcation periodic traveling-wave solutions of equation (3.3) with period (2 π) in the neighborhood of point zero by using locally method of Lyapunov-Schmidt to reduce into finite dimensional spaces. For this purpose, it is convenient to set the equation (3.3) in the form of operator equation, that is;

$$f(u,\lambda) := u^{////} - \frac{12}{r^2}u^{//} + \frac{12}{kr}uu^{//} + \frac{6}{kr}(u^{/})^2 - \frac{36(r - \frac{k^2}{r})}{k^2 r^3}u - \frac{54}{kr^3}u^2 + \frac{18}{k^2 r^2}u^3, \quad \dots (3.4)$$

where λ=(*r*, *k*) and *f* : *E* → *F* is a nonlinear Fredholm operator, *E*- Banach space of all continuous differentiable periodic functions of period (2 π), *F*- Banach space of all continuous periodic functions of period (2 π). The study of bifurcation solutions of equation (3.3) is equivalent to study the bifurcation solutions of operator equation,

$$f(u,\lambda) = 0, \quad \dots (3.5)$$

Equation (3.5) can be reduced into equivalent equation (bifurcation equation) of the form,

$$\Theta(\hat{\xi},\lambda) = 0, \quad \dots. (3.6)$$

In [4] the author shows that equation (3.6) has the form (for more detail see [4], [5]),

$$\Theta(\hat{\xi},\delta) = \begin{pmatrix} \xi_1(\xi_1^2+\xi_2^2) + 2\xi_1(\xi_3^2+\xi_4^2) + q_1(\xi_2\xi_3 - \xi_1\xi_4) + q_2\xi_1 \\ \xi_2(\xi_1^2+\xi_2^2) + 2\xi_2(\xi_3^2+\xi_4^2) - q_1(\xi_1\xi_3 + \xi_2\xi_4) + q_2\xi_2 \\ 2\xi_3(\xi_1^2+\xi_2^2) + \xi_3(\xi_3^2+\xi_4^2) + \alpha_1\xi_1\xi_2 + \alpha_2\xi_3 \\ 2\xi_4(\xi_1^2+\xi_2^2) + \xi_4(\xi_3^2+\xi_4^2) + \beta_1(\xi_1^2 - \xi_2^2) + \alpha_2\xi_4 \end{pmatrix} + \dots = 0$$

where,

$Ae_i = \tilde{\alpha}_i(\lambda)e_i$, $\hat{\xi} = (\xi_1,\xi_2,\xi_3,\xi_4)$, $\delta = (q_1,q_2,\alpha_1,\alpha_2,\beta_1)$.

$\tilde{\alpha}_i(\lambda)$ – *smooth spectral function.*

In the complex variables,

$z_1 = \xi_1 + i\xi_2$, $z_2 = \xi_3 + i\xi_4$.

Therefore, bifurcation equation can be written in the following form,

$$\begin{aligned} &z_1 z_2|z_1|^2 + 2z_1z_2|z_2|^2 + \hat{q}_1 z_1|z_2|^2 + q_2 z_1 z_2 + \dots = 0, \\ &2z_2z_1^2|z_1|^2 + z_2z_1^2|z_2|^2 + \hat{\alpha}_1(z_1^4 - |z_1|^4) + \hat{\beta}_1(z_1^4 + |z_1|^4) + \alpha_2 z_1^2 z_2 + \dots = 0. \end{aligned} \quad \dots (3.7)$$

where, $z_1, z_2 \neq 0$.

In polar coordinates $\xi_1 = r_1 \cos\theta$, $\xi_2 = r_1 \sin\theta$, $\xi_3 = r_2 \cos\varphi$, $\xi_4 = r_2 \sin\varphi$, system (3.7) is equivalent to the following system,

$$\begin{aligned} &r_1^2 + 2r_2^2 - q_1 r_2 + q_2 + \dots = 0, \\ &2r_1^2 r_2 + r_2^3 + \beta_1 r_1^2 + \alpha_2 r_2 + \dots = 0. \end{aligned} \quad \dots (3.8)$$

$$r_1,\ r_2 > 0.$$

in which we can determinate asymptotic representation of bifurcation periodic solutions. Discriminant set of system (3.8) is locally equivalent in the neighborhood of point zero to the Discriminant set of the system,

$$\begin{aligned} &r_1^2 + 2r_2^2 - q_1 r_2 + q_2 = 0, \\ &2r_1^2 r_2 + r_2^3 + \beta_1 r_1^2 + \alpha_2 r_2 = 0. \end{aligned} \quad \dots (3.9)$$

r_1, $r_2 > 0$.

To study the bifurcation periodic solutions of equation (3.3) in the neighborhood of point zero, it is sufficient to study the bifurcation solutions of system (3.9). Note that system (3.9) is the same of system (2.1) and hence all results has been found.

The point $a=\sum_{i=1}^{4}\bar{\xi}_i e_i+\Phi(\sum_{i=1}^{4}\bar{\xi}_i e_i,\lambda)$ is a solution of equation (3.5) iff $\bar{\xi}$ is a solution of equation (3.6) [9]. Thus, the existence of solutions of equation (3.5) depend on the existence of solutions of equation (3.6). From the above results we showed the existence of solutions of equation (3.6) and then there exist a bifurcation periodic solutions of equation (3.3) for which every domain has a fixed number of solutions.

References:

1- A.U. Barisovich , Proceedings of international conference, Topological methods in nonlinear analysis, Gdansk, Poland, Dec. 1-3, v.2.-8p. 1995.

2- B.V. Loginov , Tashkent: Fan, 184p., 1983.

3- E.M. Chirka , M.: Science. , 272p., 1985.

4- M.A. Abdul Hussain, J. of Basrah Researches, Vol. 32, No. 1, 49-54, 2006.

5- M. Chen, Dept. of Math., Penn State Univ., Univ. Park, PA16802, USA. 2000.

6- M.M. Vainberg, and V.A. Trenogin, M.-Science , 528p. 1969.

7- Yu.I. Sapronov , Works Dept. of Math., Voronezh Univ., V. 10.-p.82. 1973

8- Yu.I. Sapronov , Progress math. Science, T.51,V.1/-p. 101. 1969

9- Yu.I. Sapronov and V.R. Zachepa, Voronezh, 185p. 2002.

الخلاصـــة:

في هذا البحث قمنا بدراسة تحليل التفرع لنظام تكعيبي غير خطي. لقد تم اثبات وجود الحلول المنتظمة في بعض مجالات مستوي المعلمات لهذا النظام كذلك تم ايجاد المجموعة المميزة لهذا النظام. وجود و استقرارية الحلول الدورية لنظام بوسون المنتظم تم اثباتها في هذا البحث.